PERTH MODERN SCHOOL

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INDEPENDENT PUBLIC SCHOOL

Semester One Examination, 2022

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 1

Section One: Calculator-free

SOLUTIONS

Your name ____

Teacher's name____

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	6	6	50	47	35
Section Two: Calculator-assumed	11	11	100	93	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (47 Marks)**

This section has **six** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(5 marks)**

Calculate the number of integers from 1 to 168 (inclusive) which are divisible by 4 or 6.

Solution
Divisible by 4; or by 6: $\lfloor 168 \div 4 \rfloor = 42$ $\lfloor 168 \div 6 \rfloor = 28$ Divisible by 4 and 6 (12): $\lfloor 168 \div 12 \rfloor = 14$ Using the inclusion-exclusion principle: $n = 42 + 28 - 14$ $= 56$
Specific behaviours
✓ ✓ correct cases for divisible singly ✓ correct case for divisible by pairs ✓ Evidence of inclusion-exclusion principle ✓ obtains correct number of integers, no errors *(NB No marks for correct answer if no evidence of use of inclusion-exclusion principle)*

Question 2 **(8 marks)**

Consider the statement below:

If $n = 2$, then ${}^{7}\mathbf{C}_{n} = 21$.

a) **Show** that the statement is true. (2 marks)

Solution
Let $n = 2$. Then ${}^{7}\mathbf{C}_{n} = {}^{7}\mathbf{C}_{2}$ $= \dfrac{7!}{2!\,5!}$ $= 21$ Hence the statement is true.
Specific behaviours
✓ ✓ Substitutes $n = 2$ into LHS and obtains 21.

b) Write the converse of the original statement, and state whether it is true or false, giving a reason. (2 marks)

Solution
Converse: If ${}^{7}\mathbf{C}_{n} = 21$, then $n = 2$. This statement is false since the equation is also true if $n = 5$.
Specific behaviours
✓ correct converse ✓ states false with justification

c) Write the inverse of the original statement. (2 marks)

Solution
Inverse: If $n \neq 2$, then ${}^{7}\mathbf{C}_{n} \neq 21$.
Specific behaviours
✓ ✓ correct inverse

d) Write the contrapositive of the original statement, and state whether it is true or false, giving a reason. (2 marks)

Solution
Converse: Contrapositive: If ${}^{7}\mathbf{C}_{n} \neq 21$, then $n \neq 2$. The statement is true since the original statement is true.
Specific behaviours
✓ correct contrapositive ✓ states true with justification

Question 3 **(7 marks)**

Consider the vectors $\boldsymbol{a} = 9\boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{b} = 3\boldsymbol{i} - 5\boldsymbol{j}$, $\boldsymbol{c} = 6\boldsymbol{i} + 7\boldsymbol{j}$

(a) Determine the vector projection of **a** in the direction of **b**. (3 marks)

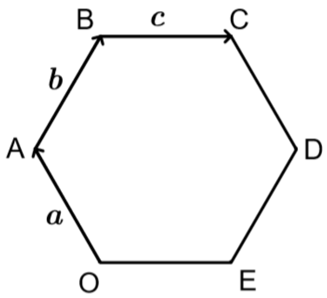
Solution
$\frac{a.b}{b.b}\boldsymbol{b} = \frac{\begin{pmatrix} 9 \\ 2 \end{pmatrix}\begin{pmatrix} 3 \\ -5 \end{pmatrix}}{\begin{pmatrix} 3 \\ -5 \end{pmatrix}\begin{pmatrix} 3 \\ -5 \end{pmatrix}}\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ $= \frac{27 - 10}{9 + 25}\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ $= \frac{17}{34}\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ $= \frac{1}{2}(3\boldsymbol{i} - 5\boldsymbol{j})$
Specific behaviours
✓ calculates scalar product ✓ calculates magnitude of b (b.b) ✓ determines vector

(b) Express $\boldsymbol{c}$ in the form $x\boldsymbol{a} + y\boldsymbol{b}$. (4 marks)

Solution
$\begin{pmatrix} 6 \\ 7 \end{pmatrix} = x\begin{pmatrix} 9 \\ 2 \end{pmatrix} + y\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ $6 = 9x + 3y \Rightarrow 2 = 3x + y$ ① $7 = 2x - 5y$ ② ① × 5 + ② $17x = 17 \Rightarrow x = 1$ $y = -1$ $\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b}$
Specific behaviours
✓ writes simultaneous equations ✓ eliminates one variable and solves ✓ solves for other variable ✓ writes in required form

Question 4 **(11 marks)**

$OABCDE$ is a regular hexagon. Let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{AB}$ and $\boldsymbol{c} = \overrightarrow{BC}$ as shown in the diagram below:



a) Write expressions for $\overrightarrow{OE}$, $\overrightarrow{ED}$ and $\overrightarrow{DC}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. (3 marks)

Solution
$\overrightarrow{OE} = \boldsymbol{c}$ $\overrightarrow{ED} = \boldsymbol{b}$ $\overrightarrow{DC} = \boldsymbol{a}$
Specific behaviours
✓ all three correct

b) Determine expressions for $\overrightarrow{AD}$ and $\overrightarrow{OB}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. (2 marks)

Solution
$\overrightarrow{AD} = \boldsymbol{b} + \boldsymbol{c} - \boldsymbol{a}$ $\overrightarrow{OB} = \boldsymbol{a} + \boldsymbol{b}$
Specific behaviours
✓ expression for $\overrightarrow{AD}$ correct ✓ expression for $\overrightarrow{OB}$ correct

c) Use a **vector method** to prove that $\overrightarrow{AD}$ is perpendicular to $\overrightarrow{OB}$.

[Hint: the internal angles in a regular hexagon are all $120°$.] (5 marks)

Solution
$\overrightarrow{AD} \cdot \overrightarrow{OB} = (\boldsymbol{b} + \boldsymbol{c} - \boldsymbol{a}) \cdot (\boldsymbol{a} + \boldsymbol{b})$ $= \boldsymbol{b} \cdot \boldsymbol{a} + \boldsymbol{b} \cdot \boldsymbol{b} + \boldsymbol{c} \cdot \boldsymbol{a} + \boldsymbol{c} \cdot \boldsymbol{b} - \boldsymbol{a} \cdot \boldsymbol{a} - \boldsymbol{a} \cdot \boldsymbol{b}$ $= \boldsymbol{b} \cdot \boldsymbol{b} + \boldsymbol{c} \cdot \boldsymbol{a} + \boldsymbol{c} \cdot \boldsymbol{b} - \boldsymbol{a} \cdot \boldsymbol{a}$ $= \lvert\boldsymbol{b}\rvert^2 - \lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{c}\rvert\lvert\boldsymbol{a}\rvert \cos 120° + \lvert\boldsymbol{c}\rvert\lvert\boldsymbol{b}\rvert \cos 60°$ $= \lvert\boldsymbol{b}\rvert^2 - \lvert\boldsymbol{a}\rvert^2 + \lvert\boldsymbol{c}\rvert\lvert\boldsymbol{a}\rvert \left(-\frac{1}{2}\right) + \lvert\boldsymbol{c}\rvert\lvert\boldsymbol{b}\rvert \frac{1}{2}$

Since $OABCDE$ is a regular hexagon, $|\boldsymbol{a}| = |\boldsymbol{b}| = |\boldsymbol{c}|$, and it follows that the above expression is equal to 0.

Hence $\overrightarrow{AD} \perp \overrightarrow{OB}$.

Specific behaviours

- ✓ uses scalar product of $\overrightarrow{AD}$ and $\overrightarrow{OB}$.
- ✓ uses expressions for $\overrightarrow{AD}$ and $\overrightarrow{OB}$ from part (b)
- ✓ expands and simplifies correctly to obtain 3rd line
- ✓ expresses $\overrightarrow{AD} \cdot \overrightarrow{OB}$ using magnitudes of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ and cosines of correct angles
- ✓ states that $|\boldsymbol{a}| = |\boldsymbol{b}| = |\boldsymbol{c}|$ and argues that $\overrightarrow{AD} \cdot \overrightarrow{OB} = 0$

Question 5 **(8 marks)**

(a) Given vectors $a = \begin{pmatrix} -16 \\ 6 \end{pmatrix}$ and $b = \begin{pmatrix} m \\ n \end{pmatrix}$ find a relationship between the components m and n if $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$. (3 marks)

Solution
If $\boldsymbol{a} = \lambda \boldsymbol{b}$ then $\begin{pmatrix} -16 \\ 6 \end{pmatrix} = \lambda \begin{pmatrix} m \\ n \end{pmatrix}$ $-16 = \lambda m$ ① $6 = \lambda n$ ② $\lambda = -\frac{16}{m} = \frac{6}{n}$ $m = -\frac{8}{3}n$
Specific behaviours
✓ Determines equation for $\boldsymbol{a}$ parallel to $\boldsymbol{b}$. ✓ Determines two expressions for λ ✓ Deduces correct relationship between m and n

(b) Two vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ are such that $\boldsymbol{p} = 3\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{q} = \boldsymbol{i} + k\boldsymbol{j}$. If the scalar projection of $\boldsymbol{p}$ in the direction of $\boldsymbol{q}$ is $\frac{1}{\sqrt{2}}$, determine the value of k, given k is an integer.

(5 marks)

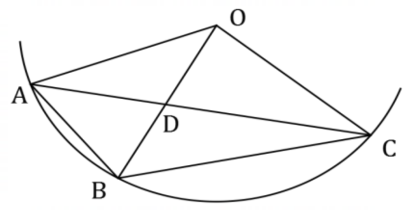
Solution
Scalar projection of $\boldsymbol{p}$ onto $\boldsymbol{q}$ is **p.q**/|**q**| i.e. $\frac{(3,2).(1,k)}{\sqrt{1+k^2}} = \frac{1}{\sqrt{2}}$ ✓ Then $3 + 2k = \sqrt{1+k^2}/\sqrt{2}$ so that $2(9 + 12k + 4k^2) = (1 + k^2)$ ✓ Hence $8k^2 + 24k + 18 - 1 - k^2 = 0$ $7k^2 + 24k + 17 = 0$ ✓ $(7k + 17)(k + 1) = 0$ and so k= -1 or $\frac{-17}{7}$ ✓ $\therefore k = -1$ ✓
Specific behaviours
✓ Correct scalar projection of $\boldsymbol{p}$ onto $\boldsymbol{q}$ formula ✓ ✓Determines the equation for k ✓ ✓Determines the two values of k

Question 6 **(8 marks)**

(a) Points A, B and C lie on an arc of a circle with centre O as shown at right.

Chord AC intersects OB at point D.

The diagram is not drawn to scale.



When $\angle ABC = 132°$ and $\angle BCA = 22°$, determine the size of $\angle ADO$. (4 marks)

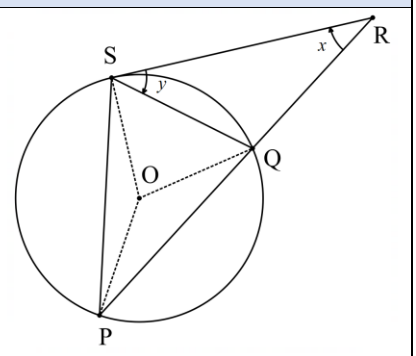
Solution
$\angle AOB = 2\angle BCA = 2 \times 22° = 44°$ $\angle BAC = 180° - 132° - 22° = 26°$ $\angle ABO = \frac{1}{2}(180° - 44°) = 68°$ $\angle ADO = 26° + 68° = 94°$
Specific behaviours
✓ obtains $\angle AOB$ ✓ obtains $\angle BAC$ ✓ obtains $\angle ABO$ ✓ correct $\angle ADO$

(b) A secant cuts a circle with centre O at points P and Q. Secant PQ is extended beyond Q to point R, where it meets a line that is a tangent to the circle at point S. Prove that $\angle QRS = \frac{1}{2}(\angle POS - \angle QOS)$ (4 marks)

Solution

Let $\angle QRS = x$ and $\angle QSR = y$. Then

$\angle PQS = x + y$ (1)
$\angle POS = 2x + 2y$ (2)
$\angle QPS = y$ (3)
$\angle QOS = 2y$ (2)



Hence

$$\frac{1}{2}(\angle POS - \angle QOS) = \frac{1}{2}(2x + 2y - 2y)$$
$$= x$$
$$= \angle QRS$$

Reasoning:
(1) sum of two interior angles is equal to the opposite exterior angle
(2) angle at centre property
(3) angle in alternate segments

Specific behaviours

✓ reasonable diagram (variations exist - secant between O and S, etc.)
✓ derives expression for $\angle AOC$
✓ derives expression for $\angle BOC$
✓ completes proof, with reasonable explanation throughout

End of questions

Supplementary page

Question number: _________